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Remark on cyclic vectors in the Dirichlet space

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Abstract

We give a new proof for the conjecture: if g is a cyclic vector in Dirichlet space and if $|f(z)| \geqslant |g(z)|$ for all points, then f is cyclic.

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1. Introduction

Let $\Delta = \{z : |z| < 1\}$ be the unit disk in the complex plane **C** and $H(\Delta)$ denote the set of all analytic functions on Δ. Lex X be a Banach space of holomorphic functions in Δ, such that the shift operator $M_z : f(z) \to zf(z)$ is a continuous map of X into itself. Given $f \in X$, we denote by $[f]$ the smallest closed M_z-invariant subspace of X containing f, namely

$$[f] = \overline{\{pf : p \textit{ is a polynomial}\}}.$$

We say that f is a cyclic vector for X if $[f] = X$.

In 1949, Beurling [2] characterized the cyclic vectors on the classical Hardy space H^2. The important consequences of this work in function theory has led many investigators to study questions related to cyclicity on other Banach spaces of analytic functions. In 1984, Brown and Shields [4] published an interesting paper giving extensive results and many

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open questions on cyclicity in Dirichlet space D. Subsequently, Brown [3] and others [6–8] have provided answers to some of these open questions. In 2006 El-fallah et al. [5] gave a new sufficient condition for $f \in D$ to be cyclic, not far from the known necessary condition. However, the main problem in [4]: Question 12 remains an open question. One of the main problems in [4] is Question 3, which say that:

If $f, g \in D$, if g is cyclic in D and if $|f(z)| \geqslant |g(z)|$ for all $z \in \Delta$, then must f be cyclic?

We know that in 1991 Richter and Sundberg confirmed this conjecture in [8]. Later, Aleman in [1] studied the invariant subspaces in general Dirichlet-type spaces and showed that this conjecture was correct again. In this paper we will give a simple proof for Question 3. Now we need some notations.

The Dirichlet space, denoted by D, defined by

$$D = \left\{ f(z) = \sum_{k \geqslant 0} a_k z^k \in H(\Delta) : \|f\|_*^2 = \sum_{k \geqslant 0} (k+1)|a_k|^2 < \infty \right\}.$$

By a direct calculation, we have $\|f\|_*^2 = \|f\|_{H^2}^2 + \|f\|_D^2$, where

$$\|f\|_{H^2}^2 = \frac{1}{2\pi} \int_0^{2\pi} |f(e^{i\theta})|^2 \, d\theta = \sum_{k \geqslant 0} |a_k|^2,$$
$$\|f\|_D^2 = \frac{1}{\pi} \int_\Delta |f'(z)|^2 \, dm(z) = \sum_{k \geqslant 0} k|a_k|^2,$$

where $d\theta$, $dm(z)$ denotes Lebesgue measure on the unit circle $\partial\Delta$, unit disc Δ respectively. It is well known that D is a Hilbert space under the norm $\|\cdot\|_*$. For more information about D see [9]. Let H^∞ be the algebra of bounded analytic functions in Δ with the norm $\|g\|_\infty = \sup_{z \in \Delta} |g(z)|$, $g \in H^\infty$.

2. Proof of Question 3

For the proof of Question 3, we need some lemmas.

Lemma 1. *For $0 < t < 1$, $z \in \Delta$, let $f_t(z) = f(tz)$. If $f \in D$, then $f_t \in D$ and $\|f_t\|_* \leqslant \|f\|_*$.*

It is easily obtained by a direct calculation.

Lemma 2. *If X is a reflexive Banach space and if $\{f_n\} \subset X$, then $f_n \to 0$ weakly if and only if both of the following conditions are satisfied*: (i) $f_n(z) \to 0\, (z \in \Delta)$, *and* (ii) $\sup \|f_n\|_X < \infty$.

See [4, Proposition 2].

Lemma 3. *If $g \in H^\infty$, $f \in D$ and $fg \in D$, then $fg \in [f]$.*

Proof. For $0 < t < 1$, $g \in H^\infty$, let P_n denote the partial sum of the power series of g_t, we can easily show that $\|P_n f - g_t f\|_* \to 0$ as $n \to \infty$ which shows that $g_t f \in [f]$. If $\sup_t \|g_t f\|_* < \infty$, then $g_t f \to gf$ weakly as $t \to 1^-$ by Lemma 2, so gf is in the weak closure of $\{pf : p \textit{ is a polynomial}\}$. Since the weak closure and norm closure of a subspace in D are same, thus $fg \in [f]$. Next we show that $\sup_t \|g_t f\|_* < \infty$.

Since $g \in H^\infty$, we have

$$\|g_t f\|_D^2 = \frac{1}{\pi}\int_\Delta |(g_t(z)f(z))'|^2 \,\mathrm{d}m(z) \leqslant 2\|g\|_\infty^2 \|f\|_D^2 + \frac{2}{\pi}\int_\Delta |f(z)g_t'(z)|^2 \,\mathrm{d}m(z).$$

Write

$$\begin{aligned}\int_D |f(z)g_t'(z)|^2 \,\mathrm{d}m(z) &\leqslant 2\int_D |f(z) - f_t(z)|^2 |g_t'(z)|^2 \,\mathrm{d}m(z) \\ &\quad + 2\int_D |f_t(z)g_t'(z)|^2 \,\mathrm{d}m(z) \\ &\triangleq 2I + 2II.\end{aligned}$$

Let $f(z) = \sum_{k \geqslant 0} a_k z^k$, it follows that

$$\begin{aligned} I &\leqslant \|g\|_\infty^2 \int_\Delta \frac{1}{(1-|tz|^2)^2} \left| \sum_{k\geqslant 0} a_k z^k - \sum_{k \geqslant 0} a_k t^k z^k \right|^2 \mathrm{d}m(z) \\ &= 2\pi \|g\|_\infty^2 \sum_{k\geqslant 0} |a_k|^2 \int_0^1 \frac{(1-t^k)^2 r^{2k}}{1-t^2r^2} r\,\mathrm{d}r \\ &\leqslant 2\pi \|g\|_\infty^2 \sum_{k\geqslant 0} |a_k|^2 \int_0^1 \frac{1-t^{2k}}{1-t^2r^2} r\,\mathrm{d}r \\ &= 2\pi \|g\|_\infty^2 \sum_{k\geqslant 0} |a_k|^2 \frac{1-t^{2k}}{2(1-t^2)} \\ &\leqslant \pi \|g\|_\infty^2 \sum_{k\geqslant 0} k|a_k|^2 < \infty. \end{aligned}$$

From Lemma 1,

$$\begin{aligned} II &\leqslant 2\int_D |(fg)_t'(z)|^2 \,\mathrm{d}m(z) + 2\int_D |f_t'(z)|^2 |g_t(z)|^2 \,\mathrm{d}m(z) \\ &\leqslant 2\|(fg)_t\|_D^2 + 2\|g\|_\infty^2 \|f_t\|_D^2 \leqslant 2\|fg\|_*^2 + 2\|g\|_\infty^2 \|f\|_*^2 < \infty. \end{aligned}$$

Thus

$$\begin{aligned} \sup_t \|g_t f\|_* &= \sup_t (\|g_t f\|_{H^2}^2 + \|g_t f\|_D^2)^{1/2} \\ &\leqslant \|g\|_\infty \|f\|_{H^2} + \sup_t \|g_t f\|_D < \infty. \qquad \square \end{aligned}$$

Theorem 1. *If $f, g \in D$, if g is cyclic in D and if $|f(z)| \geqslant |g(z)|$ for all $z \in \Delta$, then f is cyclic in D.*

Proof. If $h = g/f$, then $h \in H^\infty$ and $hf = g$, i.e. $hf \in D$. Then by Lemma 3, $g \in [f]$, which shows that f is cyclic. $\square$

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